Stationary transport in mesoscopic hybrid structures with contacts to superconducting and normal wires: A Green's function approach for multiterminal setups

Liliana Arrachea

Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Pabellón I, Ciudad Universitaria, 1428 Buenos Aires, Argentina

Departamento de Física de la Materia Condensada and BIFI, Universidad de Zaragoza, Pedro Cerbuna 12, 50009 Zaragoza, Spain

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We generalize the representation of the real-time Green's functions introduced by Langreth and Nordlander [Phys. Rev. B **43**, 2541 (1991)] and Meir and Wingreen [Phys. Rev. Lett. **68**, 2512 (1992)] in stationary quantum transport in order to study problems with hybrid structures containing normal (N) and superconducting (S) pieces without introducing Nambu representation. We illustrate the treatment in a S-N junction under a stationary bias. We derive expressions for the normal and Andreev transmission functions, and we show the equivalence between these expressions and Blonder-Tinkham-Klapwijk formulation. Finally, we investigate in detail the behavior of the equilibrium currents in a normal ring threaded by a magnetic flux with attached superconducting wires at equilibrium. We analyze the flux sensitivity of the Andreev states, and we show that their response is equivalent to the one corresponding to the Cooper pairs with momentum $q=0$ in an isolated superconducting ring.

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I. INTRODUCTION

The general framework provided by the BCS theory[1] consistently accounts for superconductivity in normal metals. Remarkably, this seems to be even true in the context of low-dimensional systems of mesoscopic scale.[2,3] BCS theory provided the basis of the seminal paper by Blonder, Tinkham, and Klapwijk (BTK).[4] In that work, the stationary transport properties of a superconductor-normal metal (S-N) junction and the subtle mechanism of the Andreev reflection leading to the effective Cooper pair tunneling through the junction were first analyzed. A similar description was followed in the study of S-N-S structure,[5–10] and was later formulated in terms of multichannel scattering matrix theory in Ref 11. BCS theory has also been the basis for the study of stationary transport in unbiased S-N-S junctions due to the Josephson effect[1,12–16] as well as the AC Josephson effect under bias.[1,16–20]

The nonequilibrium Green's function formalism[21] is a powerful technique to study quantum transport in coherent regimes. As reviewed by Rammer and Smith,[22] this formalism provides a useful framework to derive kinetic equations for normal and superconducting metals in order to describe transport in bulk materials. In the context of microscopic models for mesoscopic structures it was first introduced by Caroli *et al.*[23] and later elaborated by other authors.[24–29] That approach was also represented in the Nambu formalism to treat S-N and S-N-S junctions.[12,17,19,20] The formal equivalence between the nonequilibrium Green's function and the scattering matrix formalism to the quantum transport has been analyzed for the case of normal systems without many-body interactions under stationary[24] and time-periodic drivings.[29]

The representation of the nonequilibrium Green's functions introduced by Langreth and Nordlander[25] is particularly useful to derive compact equations for the currents along the different pieces of a mesoscopic structure.[27,28] In the present work, we employ that representation in the case of hybrid multiterminal structures containing superconducting elements that are modeled by BCS Hamiltonians.

Instead of working in Nambu's space, we derive a coupled set of Dyson's equations for the normal $\hat{G}^{R,<}_{\sigma}(\omega)$ and Gorkov's $\hat{F}^{R,<}_{\sigma}(\omega)$ retarded (R) and lesser ($<$) Green's functions. As in Refs. 26–28, we "integrate out" the degrees of freedom of the external wires (reservoirs) and, by introducing auxiliary hole propagators $\hat{\bar{g}}^{R,<}(\omega)$, we reduce the problem to solving the Dyson's equation for the usual normal Green's function with an effective self-energy. As in Refs. 26–28, the latter describes the scattering events due to the escape to the leads, but in the present case, it contains a component related to the multiscattering processes involved in the Andreev reflection. The final expressions for the currents have a compact structure that formally resemble those of Ref. 27 for normal systems.

Sections II and III are devoted to explain the theoretical treatment. We derive expressions for the currents, and we show that the transmission function of a biased system contains normal plus Andreev contributions. In Sec. IV we illustrate the approach in the simple well-known case of a two terminal setup with a linear system in contact to one normal wire and one superconducting wire under bias and we show its equivalence with BTK description. In Sec. V we employ the formalism to the study of the behavior of the equilibrium currents of a normal metallic ring threaded by a static magnetic field with several attached normal and/or superconducting wires. We address several interesting physical questions such as the minimal conditions for the development of Andreev states within the superconducting gap, the flux sensitivity of these states, and the possibility of anomalous flux quantization induced as a consequence of the proximity effect. Section VI is devoted to summary and discussion. Some technical details are presented in Appendixes A and C.

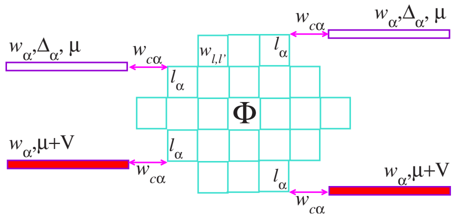

FIG. 1. (Color online) Sketch of the setup. The central grid represents the central finite system. The area enclosed by this system is threaded by a static magnetic flux Φ. The N and S wires are, respectively, indicated with open and filled lines. The arrows represent the contacts between the different systems. In each case, the parameters of the ensuing Hamiltonians are indicated.

II. THEORETICAL TREATMENT

A. Model

We introduce microscopic models for the different pieces of the setup, which consists in a finite normal system of noninteracting electrons in contact to M infinite superconducting (S) or normal (N) metallic wires (see Fig. 1). The full system is described by the following Hamiltonian:

$$H = H_{\rm cen} + \sum_{\alpha=1}^{M} (H_\alpha + H_{c\alpha}), \tag{1}$$

where H_α denote the Hamiltonians of the wires, while $H_{c\alpha}$ are the corresponding contacts establishing the connections between these systems and the central one. Although long-range superconducting order does not take place in strictly one dimension (1D), for simplicity, we consider 1D tight-binding BCS Hamiltonians with local s-wave pairing for the wires. This is a rather standard assumption (see Refs. 4–9, 12, 14, 17, 19, and 20) and the general treatment can be easily extended to multichannel wires and more general symmetries of the superconducting gap. Concretely,

$$H_\alpha = - w_\alpha \sum_{j_\alpha=1,\sigma}^{L_\alpha} (c^\dagger_{j_\alpha,\sigma} c_{j_\alpha+1,\sigma} + \text{H.c.}) - \mu_\alpha \sum_{j_\alpha=1,\sigma}^{L_\alpha} c^\dagger_{j_\alpha,\sigma} c_{j_\alpha,\sigma} + \sum_{j_\alpha=1}^{N_\alpha} (\Delta_\alpha c^\dagger_{j_\alpha,\uparrow} c^\dagger_{j_\alpha,\downarrow} + \text{H.c.}), \tag{2}$$

with $\sigma = \uparrow, \downarrow$ and being $\Delta_\alpha = 0$ for the N wires. The size of the wires approaches the thermodynamic limit ($L_\alpha \to \infty$), i.e., the wires act as macroscopic reservoirs, with well-defined chemical potential and temperature. We model the central system by a tight-binding Hamiltonian in a finite lattice of L sites with nearest-neighbor hopping. We consider the possibility of a static magnetic flux Φ threading this system, which introduces a dependence on Φ in the hopping matrix elements,

$$H_{\rm cen} = - \sum_{\langle ll' \rangle,\sigma} [w_{l,l'}(\Phi) c^\dagger_{l,\sigma} c_{l',\sigma} + \text{H.c.}] + \sum_{l=1,\sigma}^{L} \varepsilon^0_l c^\dagger_{l,\sigma} c_{l,\sigma}, \tag{3}$$

where $\langle ll' \rangle$ denotes nearest-neighbor sites. The Hamiltonians for the contacts read as follows:

$$H_{c\alpha} = - w_{c\alpha} \sum_\sigma (c^\dagger_{j_{c\alpha},\sigma} c_{l_{c\alpha},\sigma} + \text{H.c.}), \tag{4}$$

which describe hopping processes between the sites $j_{c\alpha}$ of the wires and the sites $l_{c\alpha}$ of the central system at which the wires are attached. As usual we have considered units where $\hbar = 1$.

B. Currents

The electronic current, in units of $e/\hbar$, flowing through a given bond $\langle l, l' \rangle$ of the central system is

$$J_{l,l'} = -2 \sum_\sigma \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Re}[w_{l',l}(\Phi) G^<_{l,l',\sigma}(\omega)], \tag{5}$$

while the current flowing through a given contact is

$$J_\alpha = -2 \sum_\sigma \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Re}[w_{c\alpha} G^<_{j_{c\alpha},l_{c\alpha},\sigma}(\omega)], \tag{6}$$

being

$$G^<_{l,l',\sigma}(t,t') = i \langle c^\dagger_{l\sigma}(t) c_{l'\sigma}(t') \rangle, \tag{7}$$

and $G^<_{l,l',\sigma}(\omega)$ being the corresponding Fourier transform in $t - t'$.

C. Evaluation of the Green's functions

In previous literature, the evaluation of the Green's functions for hybrid structures described in terms of tight-binding and BCS Hamiltonians has been carried out in the framework of the Nambu formalism.[12,17,19,20] We briefly present below an alternative and equivalent representation, which will allow us to analyze from a different perspective the physical processes involved in the phenomena of Andreev reflection and the development of Andreev states within the superconducting gap.

We define retarded normal and Gor'kov Green's functions as follows:

$$G^R_{j,j',\sigma}(t,t') = -i\Theta(t-t') \langle \{ c_{j,\sigma}(t), c^\dagger_{j',\sigma}(t') \} \rangle,$$

$$F^R_{j,j',\sigma}(t,t') = -i\Theta(t-t') \langle \{ c^\dagger_{j,\sigma}(t), c^\dagger_{j',\bar{\sigma}}(t') \} \rangle,$$

where $\{.,.\}$ denotes the anticommutator of the corresponding operators and $\bar{\uparrow} = \downarrow$ and $\bar{\downarrow} = \uparrow$.

It can be verified that the equations of motion for these functions are coupled and read as follows:

$$\omega G^R_{j,j',\sigma}(\omega) - \sum_{j''} \varepsilon_{j,j''} G^R_{j'',j',\sigma}(\omega) - \Delta_j F^R_{j,j',\sigma}(\omega) = \delta_{j,j'},$$

$$\omega F^R_{j,j',\sigma}(\omega) + \sum_{j''} \varepsilon_{j,j''} F^R_{j'',j',\sigma}(\omega) - \Delta^*_j G^R_{j,j',\sigma}(\omega) = 0.$$

The spacial indexes extend over the coordinates of the whole system. For coordinates on the wires $\varepsilon_{j,j'}=\Sigma_\alpha \delta_{j,j_\alpha}(\delta_{j,j'}\mu_\alpha - \delta_{j\pm 1,j'} w_\alpha)$, $\Delta_j=\Sigma_\alpha \Delta_\alpha \delta_{j,j_\alpha}$. For coordinates on the central system: $\varepsilon_{j,j'}=-\Sigma_{\langle l,l'\rangle}\delta_{j,l}\delta_{j',l'} w_{l,l'}(\Phi)$, for $\langle l,l'\rangle$, being nearest neighbors within the L-site lattice, $\varepsilon_{j,j'}=\Sigma_{l=1}^L \varepsilon^0 \delta_{l,j}\delta_{j,j'}$, and $\Delta_j=0$. For coordinates on the contacts: $\varepsilon_{j,j'} =-w_{c\alpha}(\delta_{j,l_{c\alpha}}\delta_{j',j_{c\alpha}}+\delta_{j,j_{c\alpha}}\delta_{j',l_{c\alpha}})$ and $\Delta_l=0$.

As usual, it is convenient to eliminate the degrees of freedom of the wires. Such a procedure defines self-energies for the Green's functions with coordinates belonging to what we have defined as the central system.[27,28] We summarize it in Appendix A for the present problem. The result is that the retarded Green's functions with coordinates on the central system can be expressed as elements of $L\times L$ matrices and the ensuing Dyson's equations read as follows:

$$[\hat{g}^R(\omega)]^{-1}\hat{G}^R_\sigma(\omega) + \hat{\Sigma}^{gf,R}(\omega)\hat{F}^R_\sigma(\omega) = \hat{1},$$

$$[\hat{\bar{g}}^R(\omega)]^{-1}\hat{F}^R_\sigma(\omega) + \hat{\Sigma}^{fg,R}(\omega)\hat{G}^R_\sigma(\omega) = \hat{0}, \tag{8}$$

where $\Sigma^{\nu\nu',R}_{l,l'}(\omega)=\delta_{l,l'}\Sigma_\alpha \delta_{l,l_{c\alpha}}\Sigma^{\nu\nu',R}_\alpha(\omega)$, with ν, $\nu'=g$, and f. The explicit evaluation of these functions is summarized in Appendix B. The have introduced the retarded Green's functions $\hat{g}^R(\omega)$ and $\hat{\bar{g}}^R(\omega)$, whose corresponding inverses are the following:

$$[\hat{g}^R(\omega)]^{-1} = \omega\hat{1} - \hat{\varepsilon}(\Phi) - \hat{\Sigma}^{gg,R}(\omega),$$

$$[\hat{\bar{g}}^R(\omega)]^{-1} = \omega\hat{1} + \hat{\varepsilon}(-\Phi) - \hat{\Sigma}^{ff,R}(\omega), \tag{9}$$

where $\hat{\varepsilon}(\Phi)$ contains the matrix elements of the Hamiltonian H_{cen}. In the case that all the wires are normal ($\Delta_\alpha=0, \forall\alpha$), the function $\hat{g}^R(\omega)$ is the exact retarded normal Green's function of the coupled central system, while $\Sigma^{ff,R}(\omega)=-[\Sigma^{gg,R}(-\omega)]^*$, thus $\hat{\bar{g}}^R(\omega)=[\hat{g}^R(-\omega)]^*$, which indicates that $\hat{\bar{g}}^R(\omega)$ is a propagator related to the dynamics of the holes.

The second equation (8) can be casted as follows:

$$\hat{F}^R_\sigma(\omega) = -\hat{\bar{g}}^R(\omega)\hat{\Sigma}^{fg,R}(\omega)\hat{G}^R_\sigma(\omega). \tag{10}$$

Substituting Eq. (10) in Eq. (8) the formal solution for the normal Green's is obtained as

$$[\hat{G}^R_\sigma(\omega)]^{-1} = \omega\hat{1} - \hat{\varepsilon} - \hat{\Sigma}^R_{\text{eff}}(\omega), \tag{11}$$

where we have defined an *effective normal self-energy*,

$$\hat{\Sigma}^R_{\text{eff}}(\omega) = \hat{\Sigma}^{gg,R}(\omega) + \hat{\Sigma}^{gf,R}(\omega)\hat{\bar{g}}^R(\omega)\hat{\Sigma}^{fg,R}(\omega). \tag{12}$$

The lesser counterpart of Eq. (11) is, thus, written as

$$\hat{G}^<_\sigma(\omega) = \hat{G}^R_\sigma(\omega)\hat{\Sigma}^<_{\text{eff}}(\omega)\hat{G}^A_\sigma(\omega) \tag{13}$$

being the advanced Green's function $\hat{G}^A_\sigma(\omega)=[\hat{G}^R_\sigma(\omega)]^\dagger$ Using Langreth rules,[25] $(BC)^<=B^R C^< + B^< C^A$ in the definition of Eq. (12), it can be shown that

$$\hat{\Sigma}^<_{\text{eff}}(\omega) = \hat{\Sigma}^{gg,<}(\omega) + \hat{\Sigma}^{gf,<}(\omega)\hat{\bar{g}}^A(\omega)\hat{\Sigma}^{fg,A}(\omega) + \hat{\Sigma}^{gf,R}(\omega) \times [\hat{\bar{g}}^<(\omega)\hat{\Sigma}^{fg,A}(\omega) + \hat{\bar{g}}^R(\omega)\hat{\Sigma}^{fg,<}(\omega)]. \tag{14}$$

Using the lesser counterpart of Eq. (9),

$$\hat{\bar{g}}^<(\omega) = \hat{\bar{g}}^R(\omega)\hat{\Sigma}^{ff,<}(\omega)\hat{\bar{g}}^A(\omega), \tag{15}$$

the lesser effective self-energy $\hat{\Sigma}^<_{\text{eff}}(\omega)$ can be fully expressed in terms of the bare ones, $\Sigma^{\nu\nu',<}_\alpha(\omega)=if_\alpha(\omega)\Gamma^{\nu,\nu'}_\alpha(\omega)$, with ν, $\nu'=g$, and f, which depend on the temperature T_α of the reservoirs through the Fermi function $f_\alpha(\omega)$;

$$\begin{aligned}\Sigma^<_{\text{eff},\alpha,\beta}(\omega) &= \delta_{\alpha,\beta}\Sigma^{gg,<}_\alpha(\omega) + \Lambda^R_{\alpha,\beta}(\omega)\Sigma^{fg,<}_\beta(\omega) \\ &\quad + \Sigma^{gf,<}_\alpha(\omega)\Lambda^A_{\alpha,\beta}(\omega) \\ &\quad + \sum_{\alpha'}\Lambda^R_{\alpha,\alpha'}(\omega)\Sigma^{ff,<}_{\alpha'}(\omega)\Lambda^A_{\alpha',\beta}(\omega),\end{aligned} \tag{16}$$

with $\Lambda^R_{\alpha,\beta}(\omega)=\Sigma^{gf,R}_\alpha(\omega)\bar{g}^R_{l_{c\alpha},l_{c\beta}}(\omega)$ and $\Lambda^A_{\beta,\alpha}(\omega)=[\Lambda^R_{\alpha,\beta}(\omega)]^*$. Alternatively, the above expressions can be also directly obtained after some algebra from the lesser counterpart of Eq. (8), as indicated in Appendix C.

At equilibrium, it is satisfied;

$$\Sigma^<_{\text{eff},\alpha,\beta} = if(\omega)\Gamma_{\text{eff},\alpha,\beta}(\omega), \tag{17}$$

being

$$\begin{aligned}\Gamma_{\text{eff},\alpha,\beta}(\omega) &= i[\Sigma^R_{\text{eff},\alpha,\beta}(\omega) - \Sigma^A_{\text{eff},\alpha,\beta}(\omega)] = \delta_{\alpha,\beta}\Gamma^{gg}_\alpha(\omega) \\ &\quad + \Lambda^R_{\alpha,\beta}(\omega)\Gamma^{fg}_\beta(\omega) + \Gamma^{gf}_\alpha(\omega)\Lambda^A_{\alpha,\beta}(\omega) \\ &\quad + \sum_{\alpha'=1}^{M}\Lambda^R_{\alpha,\alpha'}(\omega)\Gamma^{ff}_{\alpha'}(\omega)\Lambda^A_{\alpha',\beta}(\omega),\end{aligned} \tag{18}$$

which implies that

$$G^<_{l,l',\sigma}(\omega) = f(\omega)[G^A_{l,l',\sigma}(\omega) - G^R_{l,l',\sigma}(\omega)]. \tag{19}$$

Before closing this section, let us emphasize the formal equivalence between Eqs. (11) and (13) and the representation of Ref. 25, 27, and 28. In the present case, effective self-energies (12) and (16), however, have a more complicated structure when the leads are superconducting. In particular, they contain the normal terms $\hat{\Sigma}^{gg,R,<}(\omega)$ that represent the normal "escape to the leads" of single electrons, as well as terms involving *multiple-scattering processes*, mediated by the hole propagators $\hat{\bar{g}}^{R,<}(\omega)$. The latter acts not only locally but also extends along the different positions of the sample that are in contact to superconducting wires.

III. STATIONARY CURRENTS AND TRANSMISSION FUNCTIONS

Being able to evaluate the lesser Green's functions, we are now in the position to evaluate currents (5) and (6). We recall that a biased setup with several superconducting wires defines, in general, a time-dependent problem.[17,30] In this work we are interested in the stationary transport. Thus, in what follows we shall derive expressions for the currents in two situations: (i) a biased setup with a voltage difference between the S and the N wires, with all the S wires at the same chemical potential. In this case, currents flow through the contacts as well as along the central system. (ii) The second situation corresponds to all the wires at the same chemical potential; in which case, there are no currents flowing through the contacts and there exists only the possibility of equilibrium currents along the central structure when it is threaded by a finite magnetic flux.[31] We present below general, exact expressions for the currents, and we shall address separately the two different cases in Secs. IV and V.

Using Dyson's equation for the lesser Green's function, expressions (5) and (6) cast

$$J_{l,l'} = -2 \sum_{\sigma,\alpha,\beta=1}^{M} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \mathrm{Re}[w_{l',l}(\Phi) G^R_{l,l_{c\alpha},\sigma}(\omega) \times \Sigma^<_{\mathrm{eff},\alpha\beta}(\omega) G^A_{l_{c\beta},l',\sigma}(\omega)] \quad (20)$$

for the current along a given bond $\langle l,l' \rangle$ and

$$J_\alpha = -2 \sum_{\sigma,\alpha=1}^{M} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \mathrm{Re}[\Sigma^<_{\mathrm{eff},\alpha\beta}(\omega) G^A_{l_{c\beta},l_{c\alpha},\sigma}(\omega) + \Sigma^R_{\mathrm{eff},\alpha\beta}(\omega) G^<_{l_{c\beta},l_{c\alpha},\sigma}(\omega)] \quad (21)$$

for the current along the contact to the wire α. Details for the derivation of the latter equation from Eq. (6) follow the same lines as in Refs. 27 and 28 (see, e.g., Eq. (5) of Ref. 27, using normal Green's functions (11) and (13).

A. Equilibrium currents

When the central system is attached to wires at the same chemical potential μ, there is no charge flow through the contacts to the reservoirs. Nevertheless, if the central system is threaded by a finite magnetic flux, equilibrium currents can flow within this system. For a given bond $\langle l,l' \rangle$, the equilibrium current reads as

$$J^{\mathrm{eq}}_{l,l'} = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} f(\omega) T^{\mathrm{eq}}_{l,l'}(\omega),$$

$$T^{\mathrm{eq}}_{l,l'}(\omega) = -2\,\mathrm{Re}\{w_{l',l}(\Phi)[G^A_{l,l',\sigma}(\omega) - G^R_{l,l',\sigma}(\omega)]\} = 2\,\mathrm{Im}\left[\sum_{\sigma,\alpha,\beta=1}^{M} \Gamma_{\mathrm{eff},\alpha\beta}(\omega) w_{l',l}(\Phi) G^R_{l,l_{c\alpha},\sigma}(\omega) G^A_{l_{c\beta},l',\sigma}(\omega)\right], \quad (22)$$

where we have used equilibrium identities (17) and (19), while $\Gamma_{\mathrm{eff},\alpha\beta}(\omega)$ is defined in Eq. (18). For $\Phi=0$, the result $T^{\mathrm{eq}}_{l,l'}(\omega)|_{\Phi=0}=0$ is obtained by noticing that the function within [...] of the above expression is just the real function $-2\,\mathrm{Im}[w_{l',l}(0) G^R_{l,l'}(\omega)|_{\Phi=0}]$.

B. Nonequilibrium currents

We consider M_S S wires at $\mu_\alpha \equiv \mu$ and $M_{\mathcal{N}}=M-M_S$ N wires with a voltage difference V with respect to the superconducting ones. Following Ref. 17 we take $\mu_\alpha \equiv \mu$ in the Hamiltonians H_α for the N wires and enclose the bias V in the corresponding Fermi functions. We also consider that all the wires are at the same temperature. Therefore, for the N wires, $\Sigma^{gg,<}_\alpha(\omega)=if(\omega-V)\Gamma_\alpha(\omega)$ and $\Sigma^{ff,<}_\alpha(\omega)=if(\omega+V)\Gamma_\alpha(\omega)$, where $\Gamma_\alpha(\omega)\equiv\Gamma^{gg}_\alpha(\omega)|_{\Delta_\alpha=0}$; while for the superconducting ones, $\Sigma^{\nu\nu',<}_\alpha(\omega)=if(\omega)\Gamma^{\nu\nu'}_\alpha(\omega)$, with ν, $\nu'=g$, and f. In order to derive the expressions for the currents it is useful to express the effective lesser self-energy as follows:

$$\Sigma^<_{\mathrm{eff},\alpha,\beta}(\omega) = if(\omega)\Gamma_{\mathrm{eff},\alpha,\beta}(\omega) + i[f(\omega-V) - f(\omega)]\delta_{\alpha,\beta} \sum_{\alpha' \in \mathcal{N}} \delta_{\alpha,\alpha'}\Gamma^{gg}_{\alpha'}(\omega) + i[f(\omega+V) - f(\omega)] \sum_{\alpha' \in \mathcal{N}} \Lambda^R_{\alpha,\alpha'}\Gamma^{ff}_{\alpha'}(\omega)\Lambda^A_{\alpha',\beta}, \quad (23)$$

where $\Gamma_{\mathrm{eff},\alpha,\beta}(\omega)$ has been defined in Eq. (18).

The final expression for the nonequilibrium current along a given bond of nearest neighbors $\langle l,l' \rangle$ is

$$J_{l,l'} = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}[f(\omega-V) - f(\omega)]T(\omega). \quad (24)$$

In the case that, in addition to the bias V, the central system is threaded by a magnetic flux, we should add to the previous expression the equilibrium contribution $J^{\mathrm{eq}}_{l,l'}$ defined in Sec. III A. $J^{\mathrm{eq}}_{l,l'}$ is due to the internal currents of the single-electron orbits of the finite system, which are twisted by the static flux. Instead, the origin of the nonequilibrium contribution is a net particle flow between reservoirs through the central structure. For this reason, the nonequilibrium component depends only on the spectral properties within the energy window $[\mu,\mu+V]$, while the equilibrium one formally depends on the spectral weight of all the quantum states below μ.

The transmission function contains two contributions,

$$T_{l,l'}(\omega) = T^n_{l,l'}(\omega) + T^a_{l,l'}(\omega). \quad (25)$$

The first one is the normal transmission function,

$$T^n_{l,l'}(\omega) = 2 \sum_{\sigma,\alpha \in \mathcal{N}=1}^{M_{\mathcal{N}}} \Gamma^{gg}_\alpha(\omega) \mathrm{Im}[w_{l',l}(\Phi) G^R_{l,l_{c\alpha},\sigma}(\omega) G^A_{l_{c\alpha},l',\sigma}(\omega)], \quad (26)$$

and the second one is the Andreev transmission function,

$$T^a_{l,l'}(\omega) = 2 \sum_{\sigma,\alpha \in \mathcal{N}=1}^{M_{\mathcal{N}}} \Gamma^{ff}_\alpha(-\omega) \mathrm{Im}[w_{l',l}(\Phi) \times \bar{\Lambda}^R_{l,\alpha,\sigma}(-\omega) \bar{\Lambda}^A_{\alpha,l',\sigma}(-\omega)], \quad (27)$$

where the $\alpha \in \mathcal{N}$ denotes summation over the normal wires, while $\bar{\Lambda}^R_{l,\alpha,\sigma}(\omega)=\Sigma_\beta G^R_{l,l_{c\beta},\sigma}\Lambda^R_{\beta,\alpha}(\omega)$ and $\bar{\Lambda}^A_{\alpha,l'}(\omega)=[\bar{\Lambda}^R_{l',\alpha}(\omega)]^*$. While the normal transmission function depends on the rate at which electrons can be emitted at the normal reservoirs $\Gamma^{gg}_\alpha(\omega)$, the Andreev transmission function depends on the rate of emission of holes [we recall that $\Gamma^{ff}_\alpha(\omega)=\Gamma^{gg}_\alpha(-\omega)$]. The Andreev component depends on the multiple-scattering propagators $\bar{\Lambda}^R_{l,\alpha,\sigma}(\omega)$. Instead, the normal component depends on the usual ones $G^R_{l,l_{c\alpha},\sigma}(\omega)$. For a vanishing superconducting gap, $T^a_{l,l'}(\omega)=0$ and only the normal component survives.

Analogously, the currents through the contacts can be written as

$$J_\alpha = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}[f(\omega-V)-f(\omega)]T_\alpha(\omega), \tag{28}$$

with the transmission function also containing two components,

$$T_\alpha(\omega) = T^n_\alpha(\omega) + T^a_\alpha(\omega). \tag{29}$$

The normal transmission function reads as follows:

$$T^n_\alpha(\omega) = 2\sum_{\sigma,\beta=1}^{M}\Bigg\{\delta_{\alpha,\beta}\Gamma^{gg}_\alpha(\omega)\mathrm{Im}[G^A_{l_{c\alpha}l_{c\alpha},\sigma}(\omega)] + \sum_{\alpha'\in\mathcal{N}=1}^{M_{\mathcal{N}}}\Gamma^{gg}_{\alpha'}(\omega)\mathrm{Im}[\Sigma^R_{\alpha\beta}(\omega)G^R_{l_{c\beta}l_{c\alpha'},\sigma}(\omega)G^A_{l_{c\alpha'}l_{c\alpha},\sigma}(\omega)]\Bigg\}, \tag{30}$$

while the Andreev transmission function is

$$T^a_\alpha(\omega) = 2\sum_{\sigma,\beta=1}^{M}\sum_{\alpha'\in\mathcal{N}=1}^{M_{\mathcal{N}}}\Gamma^{ff}_{\alpha'}(-\omega)\mathrm{Im}[\Lambda^R_{\alpha,\alpha',\sigma}(-\omega) \times\Lambda^A_{\alpha',\beta,\sigma}(-\omega)G^A_{l_{c\beta}l_{c\alpha},\sigma}(-\omega)+\Sigma^R_{\mathrm{eff},\alpha\beta}(-\omega) \times\bar{\Lambda}^R_{l_{c\beta},\alpha',\sigma}(-\omega)\bar{\Lambda}^A_{\alpha',l_{c\alpha},\sigma}(-\omega)]. \tag{31}$$

IV. LINEAR BIASED SETUP WITH A SINGLE SUPERCONDUCTING WIRE AND A SINGLE NORMAL WIRE

In this section, we analyze a linear setup with a simple junction in order to benchmark the present representation for the transmission function with the one presented in the work of BTK.[4] Thus, we shall explicitly write down the previous expressions for the case of a system with two wires: one superconducting and the other one normal, which we denote, respectively, as $\alpha=N,S$. For simplicity, we also consider $\mu=0$.

In this case: $\Sigma^R_{\mathrm{eff},\alpha\beta}(\omega)=\delta_{\alpha,\beta}[\delta_{\alpha,N}\Sigma^{gg,R}_N(\omega)+\delta_{\alpha,S}\Sigma^{gf,R}_S(\omega)\bar{g}^R_{l_S,l_S}(\omega)\Sigma^{fg,R}_S(\omega)]$. The total transmission function evaluated at the contact with the N wire is $T_N(\omega)=T^n_N(\omega)+T^a_N(-\omega)$. The normal component is given by Eq. (30), which in this simple case reduces to

$$T^n_N(\omega) = \sum_\sigma \Gamma^{gg}_N(\omega)|G^R_{l_N,l_S,\sigma}(\omega)|^2\Gamma^{gg}_{\mathrm{eff},S}(\omega). \tag{32}$$

Notice that we recover the well-known structure for the normal transmission function in terms of Green's functions originally pointed out by Fisher and Lee.[24] In the present case, the function $\Gamma^{gg}_{\mathrm{eff},S}(\omega)=\Gamma^{gg}_S(\omega)-2\,\mathrm{Im}[\Sigma^{gf,R}_S(\omega)\bar{g}^R_{l_S,l_S}(\omega)\Sigma^{fg,R}_S(\omega)]$ contains the usual term $\Gamma^{gg}_S(\omega)$, which depends on the normal density of states of the superconducting lead, as well as a multiple-scattering term that depends on the hole propagator $\bar{g}^R_{l_S,l_S}(\omega)$ and the anomalous self-energy of the wire $\Sigma^{gf,R}_S(\omega)$. The Andreev transmission function reads as

$$T^a_N(\omega) = \sum_\sigma \Gamma^{ff}_N(-\omega)|\bar{\Lambda}^R_{N,N,\sigma}(-\omega)|^2\Gamma^{gg}_N(-\omega), \tag{33}$$

which actually has the formal structure of a reflection process represented in terms of Green's functions. Notice that the two coordinates of the propagator correspond to the N contact. Furthermore, the transmission function depends on the emission rate for holes in the normal wire $\Gamma^{ff}_N(\omega)=\Gamma^{gg}_N(-\omega)$ and it contains a multiple-scattering kernel,

$$\bar{\Lambda}^R_{N,N,\sigma}(\omega) = G^R_{l_N,l_S,\sigma}(\omega)\Lambda^R_{S,N}(\omega),$$

$$\Lambda^R_{S,N}(\omega) = \Sigma^{gf,R}_S(\omega)\bar{g}^R_{l_S,l_N}(\omega). \tag{34}$$

After some algebra, it can be verified that: $T^a_N(\omega)=-T^a_S(\omega)\equiv T^a(\omega)$ and $T^n_N(\omega)=-T^n_S(\omega)=T^n(\omega)$, in consistency with the continuity of the current.

The comparison with the paper of BTK (Ref. 4) becomes transparent by identifying $\omega\to E$, $T^n(\omega)\to 1-B(E)$, and $T^a(-\omega)=T^a(\omega)\to A(E)$, with $A(E)$, $B(E)$ defined in Ref. 4. A first point to notice is that we correctly recover the property $A(E)=1-B(E)$ for $E<\Delta$. In fact, within the gap $\Gamma^{\nu,\nu'}(\omega)=0$; thus,

$$\Gamma^{gg}_{\mathrm{eff},S}(\omega) = -2|\Sigma^{gf}_S(\omega)|^2\mathrm{Im}[\bar{g}^R_{l_S,l_S}(\omega)] = |\Lambda^R_{S,N}(\omega)|^2\Gamma^{ff}_N(\omega), \quad |\omega|<\Delta. \tag{35}$$

Replacing this expression in Eq. (32) we get $T^n(\omega)=T^a(\omega)$, where $|\omega|<\Delta$, in complete agreement with the result of BTK.[4]

In order to go further in the comparison we must evaluate the Green's functions. Some simple analytical expressions can be derived when the central system contains a single site with a barrier of height E_0 (equivalent to the repulsive local potential H of the work of BTK[4]) and for $\mu=0$.[32] The corresponding Hamiltonian reads as follows:

$$H_{\mathrm{cen}} = E_0 n_0, \tag{36}$$

with $n_0=\Sigma_\sigma c^\dagger_{0,\sigma}c_{0,\sigma}$. For such a system, we must consider expressions (32) and (33) with $l_S=l_N=0$. The retarded Green's function has in this case a simple form,

$$G^R_{0,0}(\omega) = \frac{1}{\omega-E_0-\Sigma_N(\omega)-\Sigma_{\mathrm{eff},S}(\omega)}, \tag{37}$$

while

$$\bar{g}_{0,0}(\omega) = \frac{1}{\omega+E_0-\Sigma^{ff}_N(\omega)-\Sigma^{ff}_S(\omega)}. \tag{38}$$

When both leads are normal, $\Delta=0$, then $\Sigma_{\mathrm{eff},S}(\omega)\to\Sigma_N(\omega)$, the transmission function contains only the normal component and reads as

$$T_0^n(\omega) = \frac{4\{\mathrm{Im}[\Sigma_N(\omega)]\}^2}{\{\omega - E_0 - 2\,\mathrm{Re}[\Sigma_N(\omega)]\}^2 + 4\{\mathrm{Im}[\Sigma_N(\omega)]\}^2}. \tag{39}$$

In analogy to the work of BTK,[4] we consider perfect matching in the hopping parameter along all the pieces; i.e., $w_\alpha = w_{c,\alpha}$ and $\alpha = N, S$. This implies $\mathrm{Re}[\Sigma_N(\omega)] = \omega/2$ and $\mathrm{Im}[\Sigma_N(\omega)] = -\Theta(|\omega| - W)\sqrt{W^2 - \omega^2}/2$, with $W = 2w_\alpha$. Also following the work of BTK,[4] we relate the normal transmission for $\Delta = 0$ in the presence of a barrier with a parameter Z,

$$T_0^n(\omega) = \frac{1}{1 + Z^2(\omega)}, \tag{40}$$

which casts $Z(\omega) = E_0/|2\,\mathrm{Im}[\Sigma_N(\omega)]|$. Notice that, unlike the model of electrons in plane waves considered by BTK,[4] in the tight-binding model the parameter Z is a function of ω. The limit $T_0^n = 1$ for $E_0 = 0$ is, however, clearly recovered. In general, the tight-binding and plane-wave models are quantitatively comparable in the limit of a large wide band, which corresponds to $\mathrm{Im}[\Sigma_N(\omega)] \sim -W/2$. In this limit the parameter Z becomes constant and $Z \sim E_0/W$. This is also in complete agreement with the interpretation of BTK (Ref. 4) since the Fermi velocity of the electrons in the tight-binding model is $v \sim 2W$.

A simple analytical expression for $T^n(\omega) = T^a(\omega)$ can be derived when $|\omega| < \Delta$ and $Z = 0$. In this case, $\mathrm{Re}[\Sigma_S^{gg}(\omega)] = \mathrm{Re}[\Sigma_S^{ff}(\omega)] = \omega\gamma(\omega)$ and $\mathrm{Re}[\Sigma_S^{gf}(\omega)] = \mathrm{Re}[\Sigma_S^{fg}(\omega)] = \Delta\gamma(\omega)$, with

$$\gamma(\omega) = \frac{1}{2}\left\{1 - \sqrt{\frac{\omega^2 - \Delta^2 - W^2}{\omega^2 - \Delta^2}}\right\}. \tag{41}$$

After some algebra we can find an explicit expression for the effective self-energy,

$$\mathrm{Re}[\Sigma_{\mathrm{eff},S}(\omega)] = -\omega\eta(\omega),$$

$$\mathrm{Im}[\Sigma_{\mathrm{eff},S}(\omega)] = -\sqrt{W^2 - \omega^2}\,\eta(\omega),$$

$$\eta(\omega) = \frac{1}{2}\left\{1 + 2\frac{\Delta^2 - \omega^2}{W^2} - \frac{2}{W^2}\sqrt{(\Delta^2 - \omega^2)(\Delta^2 + W^2 - \omega^2)}\right\}. \tag{42}$$

Substituting this expression in $G_{0,0}^R(\omega)$ with $E_0 = 0$ and T^n we obtain

$$T^n(\omega) = \frac{4\eta(\omega)}{W^2}\frac{(W^2 - \omega^2)}{[2\eta(\omega) + 1]^2}, \quad |\omega| < \Delta,$$

$$T^n(\omega) \sim 1 + \mathcal{O}\left(\frac{\Delta^2}{W^2}\right), \quad |\omega| < \Delta, \tag{43}$$

which means that even without a barrier in the junction ($E_0 = 0$) the tight-binding model presents an energy-dependent structure within the gap which is $\mathcal{O}(\frac{\Delta^2}{W^2})$. This features vanish in the wide band limit, $W \gg \Delta$, in which case, the result of BTK (Ref. 4) is exactly recovered.

For $Z \neq 0$, $\Sigma_{\mathrm{eff}}(\omega)$ as well as $G_{00}^R(\omega)$ depend on Z. The corresponding expression as well as the expression for $T^n(\omega)$ become rather cumbersome, but it can be verified that

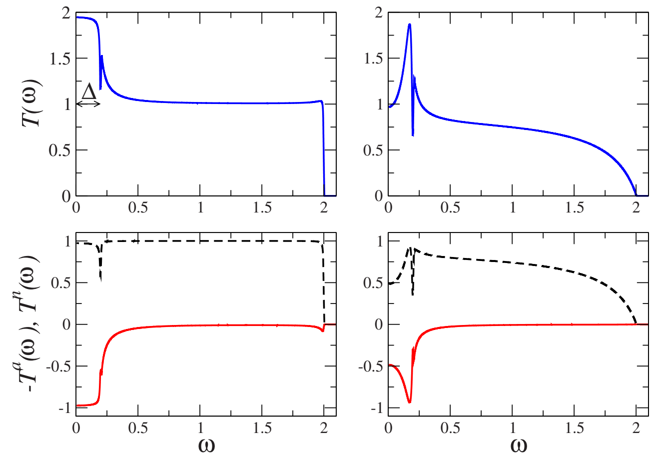

FIG. 2. (Color online) Benchmark against BTK theory. Transmission functions $T^n(\omega)$ (dashed black lines) and $-T^a(\omega)$ (red solid lines) in the lower panels and the total transmission $T(\omega) = T^n(\omega) + T^a(\omega)$ in the upper panels for a junction described by Hamiltonian (36). Left and right panels correspond to $E_0 = 0, 1$, respectively. Other parameters are $w_N = w_S = w = 1$, $\mu = 0$, and $\Delta_S = 0.2$.

$$T^n(\omega) \sim \frac{\Delta^2}{\omega^2 + (\Delta^2 - \omega^2)(1 + 2Z^2)^2} + \mathcal{O}\left(\frac{\Delta^2}{W^2}\right) \quad |\omega| < \Delta, \tag{44}$$

in complete agreement with the result of BTK (Ref. 4).

To close this comparison, numerical results for the functions $T^n(\omega)$ and $T^a(\omega)$ are shown in the lower panels of Fig. 2. We have plotted $-T^a(\omega)$ in order to be able to distinguish this function from $T^n(\omega)$ within the gap. The corresponding total transmission $T(\omega)$ is also shown in the upper panels for $E_0 = 0$ and $E_0 = 1$. The lower panels of Fig. 2 should be compared with Fig. 5 of Ref. 4. It is worth noticing, in particular, the fact that $T^a(\omega)$ is sizable within the gap and coincides with $T^n(\omega)$, while in the absence of a barrier ($E_0 = 0$), $T^a(\omega) \sim 1$. Thus $T(\omega) \sim 2$ for $|\omega| \leq \Delta$, (see upper panels of Fig. 2 and compare with Fig. 7 of Ref. 4).

V. FLUX SENSITIVITY OF THE EQUILIBRIUM CURRENTS IN A RING

We now turn to the setup without bias voltage ($V = 0$). We consider the simple case sketched in Fig. 3, where the central system corresponds to a one-dimensional ring threaded by a magnetic flux Φ, i.e., $H_{\mathrm{cen}} \equiv H_{\mathrm{ring}}$, being

$$H_{\mathrm{ring}} = -w\sum_{l=1,\sigma}^{L}(e^{-i\Phi/L}c_{l,\sigma}^\dagger c_{l+1,\sigma} + \mathrm{H.c.}) + \sum_{l=1,\sigma}^{L}\varepsilon_l^0 c_{l,\sigma}^\dagger c_{l,\sigma}, \tag{45}$$

where Φ is expressed in units of $2\pi\Phi_0$, with $\Phi_0 = e/h$ being the elementary quantum. We take the lattice constant $a = 1$, and we impose the periodic boundary condition $L + 1 \equiv 1$.

An isolated normal ring under a magnetic flux supports a persistent current with a periodicity equal to Φ_0 as a conse-

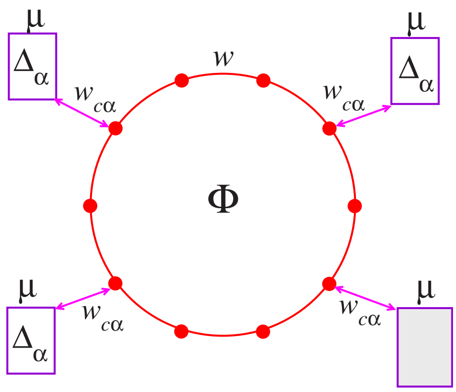

FIG. 3. (Color online) Sketch of the setup. The central system is a ring threaded by a magnetic flux in contact to superconducting and normal reservoirs at the same chemical potential μ. The only nonvanishing current is the equilibrium current along the circumference of the ring.

quence of the sensitivity of its energy levels with the threading flux. When normal metallic wires are attached to the ring, inelastic scattering effects are introduced which decrease the magnitude of this equilibrium current. However, in its qualitative behavior, in particular, the periodicity with the flux is expected to be the same as in the case of the isolated ring, provided that the inelastic scattering length $\xi_{\rm in}$ introduced by the coupling to the external wires satisfies $\xi_{\rm in} > La$. For $\xi_{\rm in} < La$, this current is, instead, expected to vanish. This is because for a short enough ring such that $\xi_{\rm in} > La$, the effect of the coupling to the wires is essentially the introduction of a finite lifetime in the energy levels, without affecting their flux sensitivity.

In the case of an isolated superconducting ring with s-wave pairing, Byers and Yang[33] showed that the periodicity of the flux-induced persistent currents is $\Phi_0/2$. This is again a consequence of the sensitivity of the energy levels—this time combined with the fact that the structure of the wave function corresponds to an ensemble of Cooper pairs instead of one of single electrons. Hybrid isolated S-N piecewise rings have been also studied, and the conclusion is that the periodicity of the persistent currents experiences a crossover between $\Phi_0/2$ and Φ_0 as the length of the superconducting piece becomes shorter than the superconducting coherence length ξ_c.[5–7]

On the other hand, a conductor between two superconductors forming a S-N-S structure is known to support Andreev states within the superconducting gap. In particular, such states are expected to develop for a ring with attached superconducting wires and it is interesting to study the flux sensitivity of these states, which should define the behavior of the equilibrium currents. It is also interesting to investigate which is the minimum number of S wires needed to develop Andreev states. Furthermore, recent studies suggest that the vortex excitations of a superconducting state can exist within a normal conductor sandwiched between two superconductors[34] due to the proximity effect. It is, therefore interesting to investigate whether it is possible that proximity effect induces also a flux periodicity of $\Phi_0/2$ in a normal ring due to the attachment to S wires.

In order to address these issues we analyze the behavior of the function $T^{\rm eq}(\omega)$. Because of the continuity of the charge, this function is independent of the bond l, with $l+1$ along the ring chosen for the evaluation of Eq. (22). Thus, the latter expression can also be written as follows:

$$T^{\rm eq}(\omega) = -\frac{2w}{L}\sum_{l=1}^{L}\sum_{\sigma,\alpha,\beta} {\rm Re}\{e^{-i\Phi/L}[G^R_{l,l+1,\sigma}(\omega) - [G^R_{l+1,l,\sigma}(\omega)]^*]\}. \tag{46}$$

In what follows, we analyze different configurations of wires.

A. Each site of the ring in contact with a wire

Let us first consider the simple case of a ring in contact to wires in a configuration that does not break the periodic translational invariance along the circumference of the ring. Such a configuration corresponds to L identical wires (N or S), each one in contact to a single site of the ring. The retarded Green's function can be easily evaluated in this case. The result is

$$G^R_{l,l',\sigma}(\omega) = \frac{1}{L}\sum_{m=0}^{L-1} e^{ik_m(l-l')} G^R_{m,\sigma}(\omega),$$

$$G^R_{m,\sigma}(\omega) = \frac{1}{\omega - \varepsilon_m(\Phi) - \Sigma^{{\rm eff},R}_m(\omega)}, \tag{47}$$

with $k_m = -\pi + 2m\pi/L$, $m=0,\ldots,L-1$, and $\varepsilon_m(\Phi) = -2w\cos(k_m + \Phi/L)$, where, for simplicity, we have taken $\mu=0$. The effective self-energy is

$$\Sigma^{{\rm eff},R}_m(\omega) = \Sigma^{gg,R}(\omega) - \Sigma^{gf,R}(\omega)\bar{g}^R_m(\omega)\Sigma^{fg,R}(\omega), \tag{48}$$

where the second term vanishes for N wires. The hole propagator of this term is

$$\bar{g}^R_m(\omega) = \frac{1}{\omega + \varepsilon_m(-\Phi) - \Sigma^{ff,R}(\omega)}. \tag{49}$$

Transforming the right hand side of Eq. (46) to the reciprocal space, it reduces to

$$T^{\rm eq}(\omega) = \frac{2}{L}\sum_{m=0}^{L-1} v_m(\Phi)\{-2\,{\rm Im}[G^R_m(\omega)]\}, \tag{50}$$

with $v_m(\Phi) = 2w\sin(k_m - \Phi/L) = \partial\varepsilon_m(\Phi)/\partial k_m$ being the velocity corresponding to the mth energy level.

In the limit where the coupling to the wires vanishes, the above expression reduces to the transmission function of an isolated ring,

$$T^{\rm eq}(\omega) \overset{w_{c\alpha}\to 0}{\to} \frac{4\pi}{L}\sum_{m=0}^{L-1} v_m(\Phi)\,\delta[\omega - \varepsilon_m(\Phi)]. \tag{51}$$

For N wires or for S wires and energies such that $|\omega| > \Delta$, a similar expression is obtained,

$$T^{\rm eq}(\omega)=\frac{4\Theta(|\omega|-\Delta)}{L}\sum_{m=0}^{L-1}\frac{v_m(\Phi)\mathrm{Im}[\Sigma_m^{{\rm eff},R}(\omega)]}{|\omega-\varepsilon_m(\Phi)-\Sigma_m^{{\rm eff},R}(\omega)|^2},\tag{52}$$

where the Θ function applies only for the case of a S wire. The above expression corresponds to a sequence of Lorentzian functions centered at energies $\sim\varepsilon_m(\Phi)+\mathrm{Re}\{\Sigma_m^{{\rm eff},R}[\varepsilon_m(\Phi)]\}$ with width $\sim\mathrm{Im}\{\Sigma_m^{{\rm eff},R}[\varepsilon_m(\Phi)]\}$. The latter parameter defines the lifetime of the levels of the ring due to the coupling to the reservoirs.

The periodicity of these currents as functions of the flux is Φ_0, which corresponds to a shift $\Phi/L=2\pi/L$ that is equivalent to a relabeling of the reciprocal points k_m. For S wires and $|\omega|<\Delta$, the functions are $\Gamma^{\nu,\nu'}(\omega)=0$, thus $\mathrm{Im}[\Sigma_m^{{\rm eff},R}(\omega)]=0$, and the only spectral contribution to $T^{\rm eq}(\omega)$ is due to the eventual development of Andreev states. The energies of these states are determined from the poles of the function $G^R_{m,\sigma}(\omega)$, which implies finding the roots of the function

$$\lambda(\omega)=\omega-\varepsilon_m(\Phi)-\mathrm{Re}[\Sigma^{gg,R}(\omega)]-\mathrm{Re}[\Sigma^{gf,R}(\omega)\Sigma^{fg,R}(\omega)]\mathrm{Re}[\bar{g}^R_m(\omega)],\tag{53}$$

where

$$\bar{g}^R_m(\omega)=\Theta(\Delta-|\omega|)\frac{1}{\omega+\bar{\varepsilon}_{-m}(\Phi)+i\eta},$$

$$\bar{\varepsilon}_m(\Phi)\sim\varepsilon_m(\Phi)+\mathrm{Re}\{\Sigma^{gg}[\varepsilon_m(\Phi)]\}..\tag{54}$$

Approximating $\mathrm{Re}[\Sigma^{\nu,\nu'}(\omega)]\sim\mathrm{Re}\{\Sigma^{\nu,\nu'}[\varepsilon_{\pm m}(\Phi)]\}$, the solution casts the following roots:

$$E_m^{\pm}(\Phi)\sim e_m^-(\Phi)\pm\sqrt{[e_m^+(\Phi)]^2+\mathrm{Re}\{\Sigma^{gf}[\varepsilon_m(\Phi)]\Sigma^{fg}[\varepsilon_m(\Phi)]\}},\tag{55}$$

with

$$e_m^{\pm}(\Phi)=\frac{\bar{\varepsilon}_m(\Phi)\pm\bar{\varepsilon}_{-m}(\Phi)}{2},\tag{56}$$

while the corresponding quasiparticle weights are

$$z_m^{\pm}=\frac{-\pi}{|\partial\lambda(\omega)/\partial\omega|_{E_m^{\pm}}}\sim\frac{-\pi}{2|E_m^{\pm}|}.\tag{57}$$

Replacing in Eq. (50), the final result for the transmission function within the superconducting gap is

$$T^{\rm eq}(\omega)=\frac{2\pi\Theta(\Delta-|\omega|)}{L}\sum_{s=\pm,m=0}^{L-1}\frac{v_m(\Phi)}{|E_m^{\pm}(\Phi)|}\delta[\omega-E_m^s(\Phi)].\tag{58}$$

For $|\omega|<\Delta$,

$$E_m^{\pm}(\Phi)\sim\beta[\varepsilon_m(\Phi)-\varepsilon_{-m}(\Phi)]\pm\sqrt{\beta^2[\varepsilon_m(\Phi)+\varepsilon_{-m}(\Phi)]^2+\gamma^2\Delta^2},\tag{59}$$

where $\beta=\{1+\gamma[\varepsilon_m(\Phi)]\}/2$, while $\gamma(\omega)$ has been defined in Eq. (41) and we are approximating $\gamma\sim\gamma[\varepsilon_m(\Phi)]$.

Remarkably, expression (58), with the energy given by Eq. (59), coincides with the expression for the persistent currents of an isolated 1D BCS tight-binding ring with hopping $2\beta w$, gap $2\gamma\Delta$, and pairs with total momentum $q=0$ (see Ref. 35). In other words, the flux sensitivity of the Andreev states in our problem is exactly the same as that observed in an isolated BCS 1D ring with pairs of momentum $q=0$. The fact that only pairs with momentum $q=0$ contribute implies that the periodicity of these currents is just the normal periodicity of a flux quantum Φ_0. These currents do not show the $\Phi_0/2$ periodicity, typical of a true superconducting ring, since the origin of that behavior is a change in $2\pi/L$ of the total momentum q of the Cooper pairs. The renormalization factor β for the hopping parameter within the ring, which determines the velocity v_m and, thus, the amplitude of the currents, depends on the superconducting coherence length of the wires, $\xi_c\sim\Delta/2w$, as well as on the tunneling ratio through the contacts, controlled by the parameter w_c. Its magnitude is large for energies close to the edge of the gap $|\varepsilon_m(\phi)|\sim\Delta$.

B. Single S wire attached to the ring

Let us now consider a single superconducting wire attached to the ring. As before, we must consider separately the contribution from states with energies within and away from the superconducting gap. To analyze the spectrum for energies $|\omega|>\Delta$, it is convenient to write the retarded Green's function as follows:

$$G^R_{l,l_{c\alpha},\sigma}(\omega)=\frac{g^0_{l,l_{c\alpha}}(\omega)}{1-\Sigma^R_{{\rm eff},\alpha}(\omega)g^0_{l_{c\alpha},l_{c\alpha}}(\omega)}\tag{60}$$

being

$$g^0_{l,l'}(\omega)=\frac{1}{L}\sum_{m=0}^{L-1}e^{-ik_m(l-l')}g^0_{k_m}(\omega),$$

$$g^0_{k_m}(\omega)=\frac{1}{\omega-\varepsilon_m(\Phi)+i\eta},$$

$$\Sigma^R_{{\rm eff},\alpha}(\omega)=\Sigma^{gg}_\alpha(\omega)+\Sigma^{gf}_\alpha(\omega)\bar{g}^0_{l_{c\alpha},l_{c\alpha}}(\omega)\Sigma^{fg}_\alpha(\omega),$$

$$\bar{g}^0_{l,l'}(\omega)=\frac{1}{L}\sum_{m=0}^{L-1}e^{-ik_m(l-l')}\bar{g}^0_{k_m}(\omega),\tag{61}$$

$$\bar{g}^0_{k_m}(\omega)=\frac{1}{\omega+\varepsilon_m(-\Phi)+i\eta}.\tag{62}$$

Substituting in Eq. (46), the transmission function reads as

$$T^{\rm eq}(\omega)=\frac{2\Theta(|\omega|-\Delta)}{L}\sum_{m=0}^{L-1}v_m(\Phi)A_m(\omega)\tag{63}$$

being

$$A_m(\omega) = \frac{\Gamma_{\text{eff},\alpha,\alpha}(\omega)|g^0_{k_m}(\omega)|^2}{|1 - \Sigma^R_{\text{eff},\alpha}(\omega)g^R_{l_{c\alpha},l_{c\alpha}}(\omega)|^2}, \tag{64}$$

which results in a Lorentzian-type profile as in the case of Eq. (52).

As in the case considered in Sec. V A, for $|\omega|<\Delta$, $\text{Im}[\Sigma^{\nu\nu',R}_\alpha(\omega)]=0$, and Andreev states can develop within the gap. In order to determine the energies of these levels, it is convenient to consider the retarded Green's functions $g^R_{l,l'}(\omega)$ and $\bar{g}^R_{l,l'}(\omega)$, defined in Eq. (9), which in the present case are the solutions of the following Dyson's equations:

$$g^R_{l,l'}(\omega) = g^0_{l,l'}(\omega) + g^R_{l,l_{c\alpha}}(\omega)\Sigma^{gg,R}_\alpha(\omega)g^0_{l_{c\alpha},l'}(\omega),$$

$$\bar{g}^R_{l,l'}(\omega) = \bar{g}^0_{l,l'}(\omega) + \bar{g}^R_{l,l_{c\alpha}}(\omega)\Sigma^{ff,R}_\alpha(\omega)\bar{g}^0_{l_{c\alpha},l'}(\omega). \tag{65}$$

Within the gap, these functions have, respectively, quasiparticle and quasihole states behaving as follows:

$$g^R_{l,l'}(\omega) \sim \frac{\Theta(|\omega|-\Delta)}{L}\sum_{m=0}^{L-1}\frac{e^{-ik_m(l-l')}z_n}{\omega - \tilde{\varepsilon}_m(\Phi) + i\eta},$$

$$\bar{g}^R_{l,l'}(\omega) \sim \frac{\Theta(|\omega|-\Delta)}{L}\sum_{m=0}^{L-1}\frac{e^{-ik_m(l-l')}z_{-n}}{\omega + \tilde{\varepsilon}_{-m}(\Phi) + i\eta}, \tag{66}$$

being $\tilde{\varepsilon}_m(\Phi) \sim \varepsilon_m(\Phi) + C\,\text{Re}\{\Sigma^{gg}_\alpha[\varepsilon_m(\Phi)]\}/L$, where $C=2$ for $\Phi=K\pi$ with K integer while $C=1$ otherwise and $z_m=-\pi\{|1 - C\partial\,\text{Re}[\Sigma^{gg,R}_\alpha(\omega)]/\partial\omega|_{\tilde{\varepsilon}_m(\Phi)}/L\}^{-1}$. In what follows, we shall approximate $z_m \sim -\pi$, which becomes exact in the limit $L\to\infty$.

The full retarded Green's function is, in turn, determined from

$$G^R_{l,l',\sigma}(\omega) = g^R_{l,l'}(\omega) + G^R_{l,l_{c\alpha},\sigma}(\omega)\Sigma^{gf,R}_\alpha(\omega)\bar{g}^R_{l_{c\alpha},l_{c\alpha}} \times(\omega)\Sigma^{fg,R}_\alpha(\omega)g^R_{l_{c\alpha},l'}(\omega). \tag{67}$$

As in the Sec. III, the ensuing solution has a quasiparticle BCS-type structure,

$$G^R_{l,l'}(\omega) \sim \frac{\Theta(|\omega|-\Delta)}{L}\sum_{s=\pm,m=0}^{L-1}\frac{e^{-ik_m(l-l')}z^s_m}{\omega - E^s_m(\Phi) + i\eta}, \tag{68}$$

with $E^\pm_m(\Phi)$ given in Eq. (59), with $\gamma\propto 1/L$ and $z^\pm_m$ given in Eq. (57).

Therefore, for a single superconducting wire connected to a large enough ring, Andreev levels tend to coincide with free particle and hole energies: $\varepsilon_m(\Phi)$ and $-\varepsilon_{-m}(\Phi)$, respectively, provided that $|\varepsilon_m(\Phi)|<\Delta$ and $|\varepsilon_{-m}(\Phi)|<\Delta$. The corresponding transmission function is formally given by Eq. (58).

In conclusion, a single superconducting wire attached to the ring generates the same qualitative behavior as L superconducting wires attached in a translational symmetrical way. But the effect is $\mathcal{O}(1/L)$ and tends to be negligible as $L\to\infty$.

VI. SUMMARY AND CONCLUSIONS

We have presented a representation of Keldysh Green's functions for stationary transport problems in systems with superconducting and normal components. As most of the relevant observables, such as the currents, depend on normal propagators, we have worked with Dyson's equations in order to eliminate the anomalous ones. This procedure has been carried out by defining auxiliary hole propagators and effective self-energies that contain multiscattering terms. In the resulting representation, the Green's functions exhibit the same structure as in normal systems. This allows for the derivation of simple and compact expressions for the currents and the transmission functions that are similar to the ones presented in Refs. 27 and 28 for normal systems.

We have presented general expressions for the currents in stationary conditions, distinguishing two situations: biased systems where transport is induced by a voltage difference and equilibrium currents induced by a static magnetic flux. In the case of biased systems, we have defined normal and Andreev transmission functions and we have compared them with results obtained in the framework of previous formalisms, in particular, the one presented by Blonder *et al.*[4]

We have finally focused on the study of the behavior of the equilibrium currents in a tight-binding normal ring with attached superconducting wires. These currents result as superpositions of the currents of all the states of the ring with energies $\varepsilon_m(\Phi)$ below the chemical potential of the wires in which electrons circulate with velocities $v_m=\partial\varepsilon_m(\Phi)/\partial k_m$.

Our main conclusions on the qualitative behavior of these currents are the following. (i) The states with energies lying away from the energy window defined by the superconducting gap present an identical qualitative behavior as those of rings attached to N wires. In particular, they have a periodicity of Φ_0 as functions of the external flux. The spectral profile related to these currents is a collection of Lorentzian functions which implies a decrease in the amplitude of the current due to inelastic scattering effects via the escape to the leads.

(ii) The states with energies within the superconducting gap of the wires behave as isolated in the sense that the spectral weight related to them consists in a collection of delta functions, indicating the lack of inelastic scattering effects. The positions of the energy levels are, however, affected by the proximity effect, and they are organized in a structure that replicates the quasiparticle spectrum of a BCS tight-binding superconducting ring with Cooper pairs of momentum $q=0$. The effective BCS tight-binding parameters are the hopping, which is the bare hopping of the ring renormalized by a factor β and a gap, which is the gap of the superconducting wires renormalized by a factor γ. The renormalizing factors depend on the superconducting coherence length of the wires and the degree of coupling between the wires and the ring. The latter effect is controlled by the strength of the coupling between these systems as well as on the number of attached wires. For a single attached wire, it is $\mathcal{O}(1/L)$ and, thus, not significant for large enough rings.

(iii) Although the proximity effect induces Andreev levels that replicate the structure of quasiparticle states of a superconducting ring within the energy window defined by the superconducting gap of the wires, these states correspond only to the subspace with winding number $q=0$. Since the periodicity in $\Phi_0/2$ of the persistent currents in superconducting rings is explained by a shift in the winding number q commensurate with the reciprocal lattice of the ring,[33,35] the restriction of the subspace with $q=0$ does not allow for such a mechanism. The consequence of this rigidity is that Andreev states have the same periodicity Φ_0 as the states of the normal ring. Let us, however, mention that the rigidity of the winding number could be due to the rigid BCS mean-field approximation considered to model the external wires. Our approach can be easily extended to more realistic wires with several channels. In such a case we should recalculate the self-energies for the leads considering, for example, a tight-binding BCS ribbon with several legs. We do not expect that such a generalization of the present geometry would lead to a different result regarding the periodicity of the equilibrium currents. In order to allow for a mechanism for the penetration of Cooper pairs with $q\neq 0$ into the ring, we should consider a more flexible model allowing for spatial fluctuations of the parameter Δ within a region of the external wires that is close to the contacts. This could eventually also permit fluctuations in the winding number q of the induced Andreev sates within the ring. A possibility to explore this mechanism is by starting from a model with an attractive two-body interaction and by treating it within a self-consistent approximation similar to that of Refs. 12 and 13 or the multichannel version of Ref. 36.

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APPENDIX A: ELIMINATING THE DEGREES OF FREEDOM OF THE RESERVOIRS

We summarize the procedure introduced in Refs. 26–28 to eliminate the degrees of freedom of the external wires in the Dyson's equation for the central system.

It is convenient to change the basis in H_α as follows:

$$c_{j_\alpha,\sigma}=\sqrt{\frac{2}{N_\alpha+1}}\sum_{n=0}^{N_\alpha}\sin(k_{n,\alpha}j_\alpha)c_{k_{n,\alpha}\sigma}, \tag{A1}$$

with $k_{n,\alpha}=n\pi/(N_\alpha+1)$ and $n=0,\ldots,N_\alpha$, which leads to

$$H_\alpha=\sum_{n=0}^{N_\alpha}\sum_\sigma\left[\varepsilon_{k_{n,\alpha}}c^\dagger_{k_{n,\alpha},\sigma}c_{k_{n,\alpha},\sigma}+\sum_{n=0}^{N_\alpha}\Delta_\alpha c^\dagger_{k_{n,\alpha},\uparrow}c^\dagger_{-k_{n,\alpha},\downarrow}+\text{H.c.}\right] \tag{A2}$$

being $\varepsilon_{k_{n,\alpha}}=-2w_\alpha\cos k_{n,\alpha}-\mu_\alpha$, and

$$H_{c,\alpha}=\sum_{n=0}^{N_\alpha}\sum_\sigma w_{\alpha,k}(c^\dagger_{k_{n,\alpha},\sigma}c_{l_{c\alpha},\sigma}+\text{H.c.}) \tag{A3}$$

being $w_{\alpha,k}=-\sqrt{\frac{2}{N_\alpha+1}}\sin k_{n,\alpha}w_{c\alpha}$.

Let us focus on the Dyson's equation with coordinates $l_{c\alpha}$ and l' belonging to the central system

$$\omega G^R_{l_{c,\alpha},l',\sigma}(\omega)-\sum_n w_{\alpha,k}G^R_{k_{n,\alpha},l',\sigma}(\omega)-\sum_{l''}\varepsilon_{l_{c,\alpha},l''}G^R_{l'',l',\sigma}(\omega)=\delta_{l_{c,\alpha},l'},$$

$$\omega F^R_{l_{c,\alpha},l',\sigma}(\omega)+\sum_n w_{\alpha,k}F^R_{k_{n,\alpha},l',\sigma}(\omega)+\sum_{l''}\varepsilon_{l_{c,\alpha},l''}F^R_{l'',l',\sigma}(\omega)=0, \tag{A4}$$

where l'' runs over all the spatial indexes of the central system while $k_{n,\alpha}$ labels degrees of freedom of the reservoir represented by H_α. The Green's functions with mixed coordinates $k_{n,\alpha}$ and l', in turn, satisfies the following equation:

$$\omega G^R_{k_{n,\alpha},l',\sigma}(\omega)-\varepsilon_{k_{n,\alpha}}G^R_{k_{n,\alpha},l',\sigma}(\omega)-w_{\alpha,k}G^R_{l_{c,\alpha},l',\sigma}(\omega)-\Delta_\alpha F^R_{k_{n,\alpha},l',\sigma}(\omega)=0,$$

$$\omega F^R_{k_{n,\alpha},l',\sigma}(\omega)+\varepsilon_{k_{n,\alpha}}F^R_{k_{n,\alpha},l',\sigma}(\omega)+w_{\alpha,k}F^R_{l_{c,\alpha},l',\sigma}(\omega)-\Delta^*_\alpha G^R_{k_{n,\alpha},l',\sigma}(\omega)=0. \tag{A5}$$

After some algebra, the above equations can be casted as follows:

$$F^R_{k_{n,\alpha},l',\sigma}(\omega)=\bar{g}^{R,0}_{k_{n,\alpha}}(\omega)[\Delta^*_\alpha G^R_{k_{n,\alpha},l',\sigma}(\omega)-w_{\alpha,k}F^R_{l_{c,\alpha},l',\sigma}(\omega)], \tag{A6}$$

$$G^R_{k_{n,\alpha},l',\sigma}(\omega)=w_{\alpha,k}[G^{R,0}_{k_{n,\alpha}}(\omega)G^R_{l_{c,\alpha},l',\sigma}(\omega)+F^{R,0}_{k_{n,\alpha}}(\omega)F^R_{l_{c,\alpha},l',\sigma}(\omega)], \tag{A7}$$

with

$$\bar{g}^{R,0}_{k_{n,\alpha}}(\omega)=\frac{1}{\omega+\varepsilon_{k_{n,\alpha}}+i\eta},$$

$$G^{R,0}_{k_{n,\alpha}}(\omega)=\frac{(\omega+\varepsilon_{kn,\alpha})}{(\omega+i\eta)^2-E^2(\varepsilon_{k_{n,\alpha}})},$$

$$F^{R,0}_{k_{n,\alpha}}(\omega)=\frac{\Delta_\alpha}{(\omega+i\eta)^2-E^2(\varepsilon_{k_{n,\alpha}})}, \tag{A8}$$

with $\eta=0^+$ and $E^2(\varepsilon_{k_{n,\alpha}})=\varepsilon^2_{k_{n,\alpha}}+\Delta^2_\alpha$.

Substituting Eq. (A6) into Eq. (A4), the latter equations can be expressed in the following way:

$$[\omega-\Sigma^{gg,R}_\alpha(\omega)]G^R_{l_{c,\alpha},l',\sigma}(\omega)+\Sigma^{gf,R}_\alpha(\omega)F^R_{l_{c,\alpha},l',\sigma}(\omega)-\sum_{l''}\varepsilon_{l_{c,\alpha},l''}G^R_{l'',l',\sigma}(\omega)=\delta_{l_{c,\alpha},l'},$$

$$[\omega - \Sigma_{\alpha}^{ff,R}(\omega)]F^{R}_{l_{c,\alpha},l',\sigma}(\omega) + \Sigma_{\alpha}^{fg,R}(\omega)G^{R}_{l_{c,\alpha},l',\sigma}(\omega) + \sum_{l''} \varepsilon_{l_{c,\alpha},l''}F^{R}_{l'',l',\sigma}(\omega) = 0. \tag{A9}$$

Notice that all the spatial indexes of the above equations run over coordinates of the central system, while the indexes corresponding to the reservoirs have been eliminated by defining the "self-energies"

$$\Sigma_{\alpha}^{\nu\nu',R}(\omega) = \sum_{n} |w_{\alpha,k}|^2 \frac{\lambda^{\nu,\nu'}(\omega,\varepsilon_{k_{n,\alpha}})}{\omega^2 - E(\varepsilon_{k_{n,\alpha}})^2} \tag{A10}$$

being $\lambda^{\nu,\nu'}(\omega,\varepsilon_{k_{n,\alpha}})=\delta_{\nu,\nu'}(\omega \pm \varepsilon_{k_{n,\alpha}})$ for $\nu=g$ and f, respectively, and $\lambda^{g,f}(\omega,\varepsilon_{k_{n,\alpha}})=[\lambda^{f,g}(\omega,\varepsilon_{k_{n,\alpha}})]^{*}=\Delta_{\alpha}$.

These steps can be repeated with each contact, which allows for the one-by-one elimination of the degrees of freedom of all the wires. The limit to the size of the wires going to infinite is summarized in Appendix B.

APPENDIX B: RETARDED SELF-ENERGIES ASSOCIATED WITH A 1D *S* WIRE

We now evaluate the spectral functions $\Gamma_{\alpha}^{\nu,\nu'}(\omega) = -2\,\mathrm{Im}[\Sigma_{\alpha}^{\nu\nu',R}(\omega)]$ corresponding to the self-energies defined in Appendix A in the thermodynamic limit, $N_{\alpha}\to\infty$. This corresponds to replacing $\Sigma_n \to (N_{\alpha}/\pi)\int_0^{\pi}dk$ in expression (A10),

$$\Gamma_{\alpha}^{\nu\nu'}(\omega) = \frac{|w_{c\alpha}|^2}{2w_{\alpha}^2}\int_{-2w_{\alpha}-\mu}^{2w_{\alpha}-\mu} du\,\lambda^{\nu,\nu'}(\omega,u)\frac{\sqrt{(2w_{\alpha})^2-(u+\mu)^2}}{E(u)} \times \{\delta[\omega - E(u)] - \delta[\omega + E(u)]\}. \tag{B1}$$

The final result is

$$\Gamma_{\alpha}^{gg}(\omega) = \Gamma_{\alpha}^{ff}(-\omega) = \mathrm{sg}(\omega)\frac{|w_{c,\alpha}|^2}{2w_{\alpha}^2}\frac{1}{r(\omega)}\{[\omega + r(\omega)]s^{+}(\omega) + [\omega - r(\omega)]s^{-}(\omega)\},$$

$$\Gamma_{\alpha}^{gf}(\omega) = [\Gamma_{\alpha}^{fg}(\omega)]^{*} = \mathrm{sg}(\omega)\frac{|w_{c,\alpha}|^2}{2w_{\alpha}^2}\frac{\Delta_{\alpha}}{r(\omega)}[s^{+}(\omega) + s^{-}(\omega)], \tag{B2}$$

with $r(\omega)=\Theta(|\omega|-|\Delta_{\alpha}|)\sqrt{\omega^2-\Delta_{\alpha}^2}$ and $s^{\pm}(\omega)=\Theta(|2w_{\alpha}|-|r(\omega)\pm\mu|)\sqrt{4w_{\alpha}^2-[r(\omega)\pm\mu]^2}$. It can be verified that, for $\mu=0$, $\Gamma_{\alpha}^{gg}(\omega)$ reduces to the Im of the diagonal component of the self-energy defined by an infinite tight-binding wire with local pairing reported in Ref. 14.

The final expressions for the retarded self-energies in the thermodynamic limit can be obtained by recourse to the Kramers-Kronig relation

$$\Sigma^{\nu\nu',R}(\omega) = \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\frac{\Gamma^{\nu,\nu'}(\omega')}{\omega - \omega' + i\eta} \tag{B3}$$

APPENDIX C: DYSON'S EQUATION FOR $\hat{G}_{\sigma}^{<}$ AND $\hat{F}_{\sigma}^{<}$

The lesser counterpart of Eq. (8) is

$$[\hat{1}\omega - \hat{\Sigma}^{gg,R}(\omega) - \hat{\varepsilon}]\hat{G}_{\sigma}^{<}(\omega) + \hat{\Sigma}^{gf,R}(\omega)\hat{F}_{\sigma}^{<}(\omega) = \hat{\Sigma}^{gg,<}(\omega)\hat{G}_{\sigma}^{A}(\omega) - \hat{\Sigma}^{gf,<}(\omega)\hat{F}_{\sigma}^{A}(\omega),$$

$$[\hat{1}\omega - \hat{\Sigma}^{ff,R}(\omega) + \hat{\varepsilon}]\hat{F}_{\sigma}^{<}(\omega) + \hat{\Sigma}^{fg,R}(\omega)\hat{G}_{\sigma}^{<}(\omega) = \hat{\Sigma}^{ff,<}(\omega)\hat{F}_{\sigma}^{A}(\omega) - \hat{\Sigma}^{fg,<}(\omega)\hat{G}_{\sigma}^{A}(\omega). \tag{C1}$$

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